On Convergence of the Horn and Schunck Optical-Flow Estimation Method

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***Abstract*—The purpose of this study is to prove convergence results for the Horn and Schunck optical-flow estimation method. Horn and Schunck stated optical-flow estimation as the minimization of a functional. When discretized, the corresponding Euler–Lagrange equations form a linear system of equations. We write explicitly this system and order the equations in such a way that its matrix is symmetric positive definite. This property implies the convergence Gauss–Seidel iterative resolution method, but does not afford a conclusion on the convergence of the Jacobi method. However, we prove directly that this method also converges. We also show that the matrix of the linear system is block tridiagonal. The blockwise iterations corresponding to this block tridiagonal structure converge for both the Jacobi and the Gauss-Seidel methods, and the Gauss–Seidel method is faster than the (sequential) Jacobi method.**

***Index Terms*—Convergence, Horn and Schunck algorithm, optical flow.**

I. INTRODUCTION

IN 1981, Horn and Schunck published Determining Optical Flow [1], one of the most referenced, influential papers in image motion analysis. Their method has been used in numerous studies and has been the benchmark for many optical-flow estimation algorithms. No convergence results were given in the original paper of Horn and Schunck and we know of none in the computer vision literature although references to the method abound [2]. Convergence is often assumed, or verified empirically.

Horn and Schunck stated optical-flow estimation as the minimization of a functional. When discretized, the corresponding Euler-Lagrange equations form a large-scale sparse system of linear equations. We write explicitly such a system and order the equations so that its matrix $\mathbf{A}$ is symmetric positive definite. This property implies the convergence of the Gauss–Seidel iterative resolution method, and of the more general relaxation methods, but does not afford a conclusion on the convergence of the Jacobi method. However, we prove directly that this method also converges. Finally, we also show that matrix $\mathbf{A}$ is block tridiagonal. The blocks of the tridiagonal decomposition have a very simple sparse structure Fig. 1. For symmetric positive definite block tridiagonal matrices, the corresponding blockwise iterations for the Jacobi and Gauss–Seidel methods converge. For this tridiagonal block decomposition, the spectral radius of the Gauss-Seidel matrix is equal to the square of the spectral radius of the Jacobi matrix, implying that the Gauss–Seidel iterations are faster that the Jacobi iterations [3], [4]. Combination of these properties suggest that the blockwise Gauss-Seidel iterations corresponding to the tridiagonal decomposition of matrix $\mathbf{A}$ are very efficient [3].

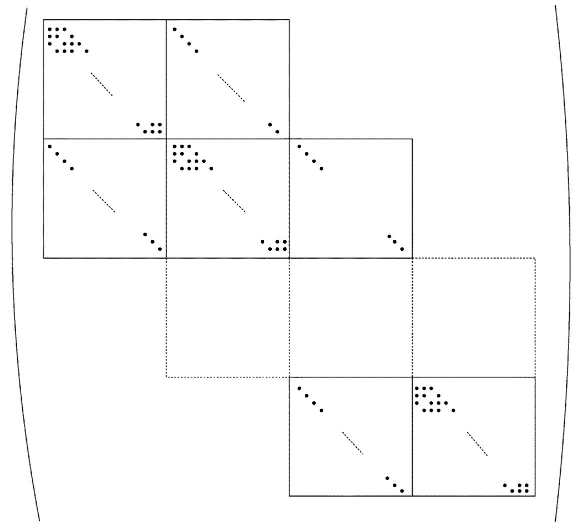

Fig. 1. Matrix $\mathbf{A}$ is block tridiagonal. For an $n \times n$ image, the blocks are of size $2n \times 2n$. The matrix is $2N \times 2N$, where $N = n^2$.

The remainder of this paper is organized as follows. In Section II, the problem is stated and the linear system of equations written. Section III deals with the property that the matrix of the linear system is symmetric positive definite, and Section IV with the property that it is block tridiagonal. Section V proves the convergence of the Jacobi iterations. Section VI contains a Conclusion.

II. PROBLEM STATEMENT

Let $I : (x, y, t) \rightarrow I(x, y, t)$ be an image sequence, where (x, y) are the spatial coordinates defined on the bounded image domain Ω, and $t \in \mathbf{R}^+$ is the time coordinate.

Horn and Schunck estimate optical flow, the field $\mathbf{W}$ of optical velocities over Ω, by minimizing the functional

$$\mathbf{W} \rightarrow E(\mathbf{W}|I) = \frac{1}{2}\int_\Omega \left(\langle \nabla I, \mathbf{W}\rangle + I_t\right)^2 dxdy + \frac{K}{2}\int_\Omega \left(\|\nabla u\|^2 + \|\nabla v\|^2\right) dxdy \quad (1)$$

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where u and v are the coordinate functions of $\mathbf{W}$, and K is a positive constant to weigh the relative contricbution of the two terms of the functional. The Euler–Lagrange equations associated with functional (1) are the following coupled partial differential equations:

$$\begin{aligned} I_x(I_x u + I_y v + I_t) - K\nabla^2 u &= 0 \\ I_y(I_x u + I_y v + I_t) - K\nabla^2 v &= 0 \end{aligned} \tag{2}$$

with the Neumann boundary conditions on the boundary $\partial\Omega$ of Ω

$$\frac{\partial u}{\partial \mathbf{n}} = 0, \quad \frac{\partial v}{\partial \mathbf{n}} = 0 \tag{3}$$

where $\partial/\partial\mathbf{n}$ is the differentiation operator in the direction of the normal $\mathbf{n}$ of boundary $\partial\Omega$. Let Ω be discretized via a unit-spacing grid D and the grid points indexed by the integers $\{1, 2, \ldots, N\}$. We take numbering to be top-down and left-to-right. For all grid point indices $i \in \{1, 2, \ldots, N\}$, a discrete approximation of the Euler–Lagrange equations (2) is

$$\begin{aligned} I_{xi}^2 u_i + I_{xi}I_{yi}v_i + I_{xi}I_{ti} - K\sum_{j\in\mathcal{N}_i}(u_j - u_i) &= 0 \\ I_{yi}I_{xi}u_i + I_{yi}^2 v_i + I_{yi}I_{ti} - K\sum_{j\in\mathcal{N}_i}(v_j - v_i) &= 0 \end{aligned} \tag{4}$$

where $(u_i, v_i) = (u, v)_i$ is the optical velocity vector at grid point i; I_{xi}, I_{yi}, I_{ti} are the values at i of I_x, I_y, I_t, respectively, and $\mathcal{N}_i$ is the set of indices of the neighbors of i. For the 8-neighborhood, card$(\mathcal{N}_i) = 8$ for points interior to the discrete image domain, and card$(\mathcal{N}_i) < 8$ for boundary points. Rewritting (4), and where $c_i = card(\mathcal{N}_i)$, we have the following system of linear equations, $i \in \{1, \ldots, N\}$:

$$(S)\begin{cases} \left(I_{xi}^2 + Kc_i\right)u_i + I_{xi}I_{yi}v_i - K\sum_{j\in\mathcal{N}_i} u_j = -I_{xi}I_{ti} \\ I_{xi}I_{yi}u_i + \left(I_{yi}^2 + Kc_i\right)v_i - K\sum_{j\in\mathcal{N}_i} v_j = -I_{yi}I_{ti}. \end{cases}$$

Let $\mathbf{z} = (z_1, \ldots, z_{2N})^t \in \mathbf{R}^{2N}$ be the vector with coordinates $z_{2i-1} = u_i$, $z_{2i} = v_i$, $i \in \{1, \ldots, N\}$, and $\mathbf{b} = (b_1, \ldots, b_{2N})^t \in \mathbf{R}^{2N}$ the vector with coordinates $b_{2i-1} = -I_{xi}I_{ti}$, $b_{2i} = -I_{yi}I_{ti}$, $i \in \{1, \ldots, N\}$. System (S) of linear equations can be written in matrix form as

$$\mathbf{Az} = \mathbf{b} \tag{5}$$

where $\mathbf{A}$ is the $2N \times 2N$ matrix with elements $\mathbf{A}_{2i-1,2i-1} = I_{xi}^2 + Kc_i$, $\mathbf{A}_{2i,2i} = I_{yi}^2 + Kc_i$, $\mathbf{A}_{2i-1,2i} = \mathbf{A}_{2i,2i-1} = I_{xi}I_{yi}$ for all $i \in \{1, \ldots, N\}$, and $\mathbf{A}_{2i-1,2j-1} = \mathbf{A}_{2i,2j} = -K$, for all $i, j \in \{1, \ldots, N\}$ such that $j \in \mathcal{N}_i$, all other elements being equal to zero.

System (S) is a large-scale sparse system of linear equations. Such systems are best solved by iterative methods [3], [4]. Assuming matrix $\mathbf{A}$ is nonsingular, the problem is to solve (5) by an efficient convergent method. In the following, we prove that matrix $\mathbf{A}$ has properties that lead to such methods. We will assume that $\mathbf{A}$ is nonsingular.

III. Matrix $\mathbf{A}$ Is Positive Definite

One can easily verify that matrix $\mathbf{A}$ is symmetric. Matrix $\mathbf{A}$ is also positive definite. To show this we verify that $\mathbf{z}^t\mathbf{Az} > 0$ for all $\mathbf{z} \in \mathbf{R}^{2N}$, $\mathbf{z} \neq \mathbf{0}$. We have

$$\begin{aligned} \mathbf{z}^t\mathbf{Az} &= \sum_{i=1}^N \left(\left(I_{xi}^2 + Kc_i\right)u_i + I_{xi}I_{yi}v_i - K\sum_{j\in\mathcal{N}_i} u_j\right)u_i \\ &\quad + \sum_{i=1}^N \left(I_{xi}I_{yi}u_i + \left(I_{yi}^2 + Kc_i\right)v_i - K\sum_{j\in\mathcal{N}_i} v_j\right)v_i \\ &= \sum_{i=1}^N \Bigg(\left(I_{xi}^2u_i^2 + I_{yi}^2v_i^2 + 2I_{xi}I_{yi}u_iv_i\right) + Kc_i\left(u_i^2 + v_i^2\right) \\ &\qquad - K\sum_{j\in\mathcal{N}_i} u_ju_i - K\sum_{j\in\mathcal{N}_i} v_jv_i\Bigg) \\ &= \sum_{i=1}^N (I_{xi}u_i + I_{yi}v_i)^2 \\ &\quad + K\sum_{i=1}^N \left(c_i\left(u_i^2 + v_i^2\right) - \sum_{j\in\mathcal{N}_i} u_ju_i - \sum_{j\in\mathcal{N}_i} v_jv_i\right). \end{aligned}$$

We also have

$$\begin{aligned} &\sum_{i=1}^N \left(c_i\left(u_i^2 + v_i^2\right) - \sum_{j\in\mathcal{N}_i} u_ju_i - \sum_{j\in\mathcal{N}_i} v_jv_i\right) \\ &\quad = \sum_{i=1}^N \Bigg(\sum_{j\in\mathcal{N}_i, j>i}\left(u_i^2 + u_j^2 - 2u_ju_i\right) \\ &\qquad + \sum_{j\in\mathcal{N}_i, j>i}\left(v_i^2 + v_j^2 - 2v_jv_i\right)\Bigg) \\ &\quad = \sum_{i=1}^N \sum_{j\in\mathcal{N}_i; j>i}\left((u_i - u_j)^2 + (v_i - v_j)^2\right) \end{aligned}$$

and therefore

$$\begin{aligned} \mathbf{z}^t\mathbf{Az} &= \sum_{i=1}^N (I_{xi}u_i + I_{yi}v_i)^2 \\ &\quad + K\sum_{i=1}^N \sum_{j\in\mathcal{N}_i; j>i}\left((u_i - u_j)^2 + (v_i - v_j)^2\right). \end{aligned} \tag{6}$$

For $\mathbf{z} \neq \mathbf{0}$, we have $\mathbf{z}^t\mathbf{Az} = 0$ if and only if the terms in both sums on the right-hand side of (6) are zero. The terms in the second sum are zero if and only if $\mathbf{W}$ is constant over D. The terms in the first sum are zero if and only if this constant optical velocity vector is orthogonal to the image spatial gradient at all points in D. In such a case, and because the spatial gradient is orthogonal at every point to the level curve $I = constant$ through this point, the optical velocity vector is in the direction of the level curve at every point. Excluding this special case where $\mathbf{W}$ is constant over D and in the direction of constant intensity lines,[1] we have $\mathbf{z}^t\mathbf{Az} > 0$ for $\mathbf{z} \neq \mathbf{0}$ and $\mathbf{A}$ is positive definite.

[1]In the continuous case this means that I does not change in time and that its graph is a plane.

Because $\mathbf{A}$ is a positive definite matrix, the pointwise and blockwise Gauss-Seidel and relaxation iterative methods for solving the system of linear (5) converge. This is a standard result in numerical linear algrebra and a proof can be found in numerical analysis textbooks such as [3] and [4]. However, the property that matrix $\mathbf{A}$ is positive definite does not allow us to conclude that Jacobi iterations converge. Jacobi iterations for a 2×2 block division of matrix $\mathbf{A}$ (which gives an $N \times N$ block matrix) are often used with the Horn and Schunck optical-flow estimation method [1]. These iterations are, for all $i \in \{1, \dots, N\}$

$$\begin{aligned} u_i^{k+1} &= \frac{I_{yi}^2 + Kc_i}{c_i\left(I_{xi}^2 + I_{yi}^2\right) + Kc_i^2} \sum_{j \in \mathcal{N}_i} u_j^k \\ &\quad - \frac{I_{xi} I_{yi}}{c_i\left(I_{xi}^2 + I_{yi}^2\right) + Kc_i^2} \sum_{j \in \mathcal{N}_i} v_j^k \\ &\quad - \frac{I_{xi} I_{ti}}{I_{xi}^2 + I_{yi}^2 + Kc_i} \\ v_i^{k+1} &= \frac{-I_{xi} I_{yi}}{c_i\left(I_{xi}^2 + I_{yi}^2\right) + Kc_i^2} \sum_{j \in \mathcal{N}_i} u_j^k \\ &\quad + \frac{I_{xi}^2 + Kc_i}{c_i\left(I_{xi}^2 + I_{yi}^2\right) + Kc_i^2} \sum_{j \in \mathcal{N}_i} v_j^k \\ &\quad - \frac{I_{yi} I_{ti}}{I_{xi}^2 + I_{yi}^2 + Kc_i.} \end{aligned} \tag{7}$$

With the Jacobi method, the update is done for all points of the image domain and the updated values are used at the next iteration. With the Gauss–Seidel method, the updated values are used are soon as they are available. For the 2×2 block division of matrix $\mathbf{A}$, the Gauss–Seidel iterations are, for all $i \in \{1, \dots, N\}$

$$\begin{aligned} u_i^{k+1} &= \frac{I_{yi}^2 + Kc_i}{c_i\left(I_{xi}^2 + I_{yi}^2\right) + Kc_i^2} \left(\sum_{j \in \mathcal{N}_i; j<i} u_j^{k+1} + \sum_{j \in \mathcal{N}_i; j>i} u_j^k \right) \\ &\quad - \frac{I_{xi} I_{yi}}{c_i\left(I_{xi}^2 + I_{yi}^2\right) + Kc_i^2} \left(\sum_{j \in \mathcal{N}_i; j<i} v_j^{k+1} + \sum_{j \in \mathcal{N}_i; j>i} v_j^k \right) \\ &\quad - \frac{I_{xi} I_{ti}}{I_{xi}^2 + I_{yi}^2 + Kc_i} \\ v_i^{k+1} &= \frac{-I_{xi} I_{yi}}{c_i\left(I_{xi}^2 + I_{yi}^2\right) + Kc_i^2} \left(\sum_{j \in \mathcal{N}_i; j<i} u_j^{k+1} + \sum_{j \in \mathcal{N}_i; j>i} u_j^k \right) \\ &\quad + \frac{I_{xi}^2 + Kc_i}{c_i\left(I_{xi}^2 + I_{yi}^2\right) + Kc_i^2} \left(\sum_{j \in \mathcal{N}_i; j<i} v_j^{k+1} + \sum_{j \in \mathcal{N}_i; j>i} v_j^k \right) \\ &\quad - \frac{I_{yi} I_{ti}}{I_{xi}^2 + I_{yi}^2 + Kc_i}. \end{aligned} \tag{8}$$

As is well known, the significant differences between iterations (9) and (8) are the following: with the Jacobi iterations (8), (u_i^{k+1}, v_i^{k+1}), $i = 1, \dots, 2N$ at iteration $k+1$ are computed using the values (u_i^k, v_i^k), $i = 1, \dots, 2N$ of the preceding iteration k. Therefore, $4N$ values must be stored. With the Gauss–Seidel iterations (9), only $2N$ values need storage. However, with the Jacobi method, the updates can be activated in parallel for all image points at each iteration, to afford a significant gain in execution time.

IV. Matrix $\mathbf{A}$ Is Block Tridiagonal

The Gauss–Seidel method, and the more general relaxation methods, to solve the system of linear (5) will converge blockwise for any block division of matrix $\mathbf{A}$ with square diagonal blocks. For an $M \times M$ block matrix, block iterative methods involve solving, at each iteration, M linear systems of equations whose matrices are the submatrices of $\mathbf{A}$ corresponding to the diagonal blocks. Although generally faster than corresponding pointwise methods, block methods are to be used only when the increase in the time of an iteration due to solving these M linear systems is sufficiently offset by the accelation in convergence. For system (S), there is a remarkable block division which makes matrix $\mathbf{A}$ block tridiagonal. Combined with the property that $\mathbf{A}$ is symmetric positive definite, this characteristic affords efficient resolution of the corresponding linear system [3]. For an $n \times n$ discrete image, the blocks are $2n \times 2n$. The block tridiagonal form is due to the fact that points with index αn, $1 \leq \alpha \leq n$ do not have a neighbor on the right, and those with index $\beta n + 1$, $0 \leq \beta \leq n-1$ do not have a neighbor on the left. The tridiagonal structure of $\mathbf{A}$ for the 4-neighborhood is illustrated in Fig. 1.

For a symmetric positive definite matrix that is block tridiagonal, the corresponding blockwise iterations, i.e., the iterations corresponding to the tridiagonal block decomposition Fig. 1, converge for the Jacobi, the Gauss–Seidel, and the relaxation methods. The spectral radius of the Gauss–Seidel matrix is equal to the square of the spectral radius of the Jacobi matrix, implying that the Gauss–Seidel iterations are faster that the Jacobi iterations. The relaxation methods (for relaxation factors strictly between 0 and 2) are faster than the Gauss–Seidel method, and there exists a unique optimal relaxation factor ([3] and [4]).

V. The Jacobi Iterations

The fact that matrix $\mathbf{A}$ is symmetric positive definite, and block tridiagonal, does not afford a conclusion on the convergence of the Jacobi 2×2 block iterations (8). However, we will prove directly that these iterations converge. Iterations (8) can be written in matrix form

$$\mathbf{z}^{k+1} = \mathbf{P}\mathbf{z}^k + \mathbf{d} \tag{9}$$

where $\mathbf{z} \in \mathbf{R}^{2N}$ is the vector with entry $2i-1$ equal to u_i and entry $2i$ equal to v_i, $\forall i \in \{1, \dots, N\}$; $\mathbf{d}$ is the $2N$ real vector with entries $\mathbf{d}_{2i-1} = -(I_{xi} I_{ti} / (I_{xi}^2 + I_{yi}^2 + Kc_i))$, $\mathbf{d}_{2i} = -(I_{yi} I_{ti} / (I_{xi}^2 + I_{yi}^2 + Kc_i))$, $i \in \{1, \dots, N\}$; and $\mathbf{P}$ is the $2N \times 2N$ Jacobi matrix corresponding to the 2×2 block division of $\mathbf{A}$. The elements of $\mathbf{P}$ are:[2] $\mathbf{P}_{2i,2i} = \mathbf{P}_{2i-1,2i-1} = 0, \forall i \in \{1, \dots, N\}$

$$\begin{aligned} \mathbf{P}_{2i-1,2j-1} &= \frac{\left(I_{yi}^2 + Kc_i\right)}{c_i\left(I_{xi}^2 + I_{yi}^2\right) + Kc_i^2} \\ \mathbf{P}_{2i-1,2j} &= \mathbf{P}_{2i,2j-1} \\ &= -\frac{I_{xi} I_{yi}}{c_i\left(I_{xi}^2 + I_{yi}^2\right) + Kc_i^2} \\ \mathbf{P}_{2i,2j} &= \frac{\left(I_{xi}^2 + Kc_i\right)}{c_i\left(I_{xi}^2 + I_{yi}^2\right) + Kc_i^2} \\ &\forall i, j \in \{1, \dots, N\}, \text{ such that } j \in \mathcal{N}_i \end{aligned}$$

[2]Recall (from [3] or [4], for instance) that $\mathbf{P} = \mathbf{I} - \mathbf{D}^{-1}\mathbf{A}$ and $\mathbf{d} = \mathbf{D}^{-1}\mathbf{b}$, where $\mathbf{I}$ is the indentity matrix and $\mathbf{D}$ is the (nonsingular) block diagonal matrix whose diagonal blocks are the 2×2 diagonal block submatrices of $\mathbf{A}$.

with all other elements equal to zero, including the diagonal entries. We consider the following norm on vectors $\mathbf{z} = (z_i)_i \in \mathbb{R}^{2N}$:

$$\mathbf{z} \mapsto \|\mathbf{z}\| = \max_{i=1,\ldots,N} \left(z_{2i-1}^2 + z_{2i}^2\right)^{\frac{1}{2}}. \tag{10}$$

This is indeed a norm for vectors in $\mathbb{R}^{2N}$. One can verify immediately that the function defined by (10) is nonnegative ($\|\mathbf{z}\| \geq 0$), positive ($\|\mathbf{z}\| = 0 \leftrightarrow \mathbf{z} = \mathbf{0}$), and homogeneous ($\|c\mathbf{z}\| = |c|\|\mathbf{z}\|$). That it satisfies the triangle inequality can be shown as follows. Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in $\mathbb{R}^{2N}$. Because the Euclidean norm in $\mathbb{R}^2$ satisfies the triangle inequality, we have, for each $i \in \{1, \ldots, N\}$

$$\begin{aligned}\left((x_{2i-1} + y_{2i-1})^2 + (x_{2i} + y_{2i})^2\right)^{\frac{1}{2}}& \\ \leq \left(x_{2i-1}^2 + x_{2i}^2\right)^{\frac{1}{2}} + \left(y_{2i-1}^2 + y_{2i}^2\right)^{\frac{1}{2}}&. \end{aligned}\tag{11}$$

Therefore, we have

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\| &= \max_{i=1,\ldots,N} \left((x_{2i-1}+y_{2i-1})^2+(x_{2i}+y_{2i})^2\right)^{\frac{1}{2}} \\ &\leq \max_{i=1,\ldots,N} \left(\left(x_{2i-1}^2+x_{2i}^2\right)^{\frac{1}{2}}+\left(y_{2i-1}^2+y_{2i}^2\right)^{\frac{1}{2}}\right) \\ &\leq \max_{i=1,\ldots,N} \left(x_{2i-1}^2+x_{2i}^2\right)^{\frac{1}{2}}+\max_{i=1,\ldots,N} \left(y_{2i-1}^2+y_{2i}^2\right)^{\frac{1}{2}} \\ &= \|\mathbf{x}\|+\|\mathbf{y}\|.\end{aligned}$$

Now let $\mathbf{z} \in \mathbb{R}^{2N}$ and for $i = 1, \ldots, N$ let $\mathbf{z}^i = (z_j^i)_j \in \mathbb{R}^{2N}$ be defined by $z_{2j-1}^i = z_{2j-1}$ and $z_{2j}^i = z_{2j}$ for $j \in \mathcal{N}_i$, all other elements being equal to zero. We have

$$\begin{aligned}\|\mathbf{P}\mathbf{z}\| &= \max_{i=1,\ldots,N} \left(\left(\sum_{j=1}^{2N} P_{(2i-1)j} z_j\right)^2 + \left(\sum_{j=1}^{2N} P_{(2i)j} z_j\right)^2\right)^{\frac{1}{2}} \\ &= \max_{i=1,\ldots,N} \left(\left(\sum_{j=1}^{2N} P_{(2i-1)j} z_j^i\right)^2 + \left(\sum_{j=1}^{2N} P_{(2i)j} z_j^i\right)^2\right)^{\frac{1}{2}} \\ &\leq \max_{i=1,\ldots,N} \left(\left(\sum_{j=1}^{2N} P_{(2i-1)j}^2\right)\left(\sum_{j=1}^{2N} \left(z_j^i\right)^2\right) + \left(\sum_{j=1}^{2N} P_{(2i)j}^2\right)\left(\sum_{j=1}^{2N} \left(z_j^i\right)^2\right)\right)^{\frac{1}{2}} \\ &\leq \max_{i=1,\ldots,N} \left(\sum_{j=1}^{2N} P_{(2i-1)j}^2 + \sum_{j=1}^{2N} P_{(2i)j}^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{2N} \left(z_j^i\right)^2\right)^{\frac{1}{2}} \\ &\leq \max_{i=1,\ldots,N} \left(\sum_{j=1}^{2N} \left(P_{(2i-1)j}^2 + P_{(2i)j}^2\right)\right)^{\frac{1}{2}} \\ &\quad \times \left(c_i \max_{j=1,\ldots,N} \left(\left(z_{2j-1}^i\right)^2 + \left(z_{2j}^i\right)^2\right)\right)^{\frac{1}{2}} \\ &\leq \max_{i=1,\ldots,N} \sqrt{c_i} \left(\sum_{j=1}^{2N} \left(P_{(2i-1)j}^2 + P_{(2i)j}^2\right)\right)^{\frac{1}{2}} \|\mathbf{z}^i\|.\end{aligned}$$

All of the nonzero elements of $\mathbf{z}^i$ are elements of $\mathbf{z}$. Therefore, $\|\mathbf{z}^i\| \leq \|\mathbf{z}\|$ and

$$\|\mathbf{P}\mathbf{z}\| \leq \max_{i=1,\ldots,N} \sqrt{c_i} \left(\sum_{j=1}^{2N} \left(P_{(2i-1)j}^2 + P_{(2i)j}^2\right)\right)^{\frac{1}{2}} \|\mathbf{z}\|. \tag{12}$$

Because

$$\sum_{j=1}^{2N} \left(P_{(2i-1)j}^2 + P_{(2i)j}^2\right) = \frac{1}{c_i^2}\left(1 + \frac{(Kc_i)^2}{\left(I_{xi}^2 + I_{yi}^2 + Kc_i\right)^2}\right) \quad i = 1, \ldots, N \tag{13}$$

we have

$$\|\mathbf{P}\mathbf{z}\| \leq \max_{i=1,\ldots,N} \frac{1}{\sqrt{c_i}} \left(1 + \frac{(Kc_i)^2}{\left(I_{xi}^2 + I_{yi}^2 + Kc_i\right)^2}\right)^{\frac{1}{2}} \|\mathbf{z}\|. \tag{14}$$

When $c_i > 2$ for all i (with the 8-neighborhood, for instance), we have

$$\frac{1}{\sqrt{c_i}} \left(1+\frac{(Kc_i)^2}{\left(I_{xi}^2+I_{yi}^2+Kc_i\right)^2}\right)^{\frac{1}{2}} \leq \frac{\sqrt{2}}{\sqrt{c_i}} < 1 \quad i = 1, \ldots, N. \tag{15}$$

Therefore

$$\|\mathbf{P}\mathbf{z}\| \leq \alpha\|\mathbf{z}\|, \quad \text{with} \quad 0 < \alpha < 1. \tag{16}$$

Let $\mathbf{z}^*$ be the solution of system (5). It satisfies the relation (see footnote for a quick verification)

$$\mathbf{z}^* = \mathbf{P}\mathbf{z}^* + \mathbf{d}. \tag{17}$$

From relations (9), (16), and (17), we get

$$\|\mathbf{z}^{k+1} - \mathbf{z}^*\| = \left\|\mathbf{P}(\mathbf{z}^k - \mathbf{z}^*)\right\| \leq \alpha\|\mathbf{z}^k - \mathbf{z}^*\| \tag{18}$$

which shows that the Jacobi iterations (9) converge.

VI. Conclusion

The goal of this study was to prove convergence results for the Horn and Schunck optical-flow estimation method. We wrote the linear system of equations in this method explicitly, ordering the equations so that its matrix is symmetric positive definite, which allowed us to conclude that the iterative pointwise and blockwise Gauss–Seidel and relaxation methods converge. We also showed that the matrix is block tridiagonal, which allowed to draw a conclusion on the convergence of block Jacobi, Gauss–Seidel, and relaxations methods corresponding to the tridiagonal block decomposition of the matrix. Finally, we proved directly that the Jacobi iterations converge. The analysis showed that the Gauss–Seidel or relaxation methods should be used rather that the Jacobi iterations, unless a parallel implementation is considered for the Jacobi method.

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